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SURFACE STRESS IN SOLIDS

MORTON E. GURTIN
Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

and

A. IAN MURDOCH
Department of Engineering, Science, University of Cincinnati, Cincinnati, OH 45221, U.S.A.

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Abstract—In a recent paper[6] a general theory of surface stress was presented. Here we discuss several simple solutions within this theory.

1. INTRODUCTION

Certain crystals, when cleaved, exhibit a surface stress which gives rise to small but detectable strains in the interior of the crystal (see [1]). More generally, microscopic considerations predict the presence of surface stress whenever a new surface is created (see [2–5]). The small strain magnitudes attributable to surface stress together with the sensitivity of this stress to the environment render experimental difficulties. Further, the absence of a general continuum theory of surface stress has resulted in problems concerning the analysis and interpretation of experimental data.

In a recent paper[6] a general mathematical theory of surface stress was presented; within the linearized version of this theory the surface stress tensor is the sum of a residual stress tensor and a linear function of surface strain[2, 3]. This clearly generalizes the concept of a surface tension, accommodating situations in which the surface is anisotropic and/or supports compressive stress as predicted for certain crystals on the basis of atomistic calculations[2].

The presence of surface stress results in a non-classical boundary condition which gives the surface traction on the substrate in terms of surface stress and inertia. This boundary condition and the surface stress–strain relation together with the equations of classical elasticity (to be satisfied within the body) form a coupled system of field equations. It is the purpose of this paper to exhibit some simple solutions of these equations and to discuss their physical significance. In this respect it is to be remarked that the theory outlined above also models situations in which the surface of a body is coated with a thin layer of another material. Currently much attention has been devoted to such films since they have applications as surface wave-guides (see [7, 8]). The theory may also approximate the behavior of bodies whose surfaces have been "shot-peened"; that is, bombarded with small pellets, so forming, on the surface, a thin skin in which the stress is compressive (see [9]).

In Section 2 are presented the basic equations for bodies and their surfaces assuming isotropy and homogeneity. The magnitudes of the physical parameters involved are tabulated for an iron free surface and a very thin iron film upon a glass substrate. Several solutions within the equilibrium theory appear in Section 3. These model situations in which a part of the body is removed, thereby exposing a free surface. Although the body may be initially free of stress, the residual stress located in the newly-created surface generates a stress field in the body. The results quoted are those for an infinite circular cylinder[6], a sphere[10], and an infinite body of square cross section. The last problem was first considered by Herring[3], who established some order-of-magnitude estimates; here the outcome of a study by Dunham and Gurtin[11] is discussed.

The possibility of compressive surface stress led Orowan[5] to conjecture that "a strong negative surface tension may conceivably give rise to buckling, a wrinkling of the outermost atomic planes..." The last problem we discuss in Section 3 addresses itself to Orowan's conjecture. We there present results of Andreussi and Gurtin[12] which show that surface buckling is possible whenever either the residual surface stress is compressive or the surface stress–strain modulus is negative.

Section 4 is concerned with dynamical problems; in particular, vibrations of a thin beam and acoustic wave propagation in a half-space. The beam problem, motivated by experiments of Lagowski, Gatos and Sproles[13], was studied by Gurtin, Markenscoff and Thurston[14] who showed that a constant surface tension has no effect on the natural frequency, thus contradicting the assertions of [13]. Both bulk and surface plane harmonic waves in a half-space are discussed, details being drawn from work on the reflection of bulk waves at a plane surface by Gurtin and Murdoch[15] and on the propagation of Rayleigh and Love waves by Murdoch[16].

Further considerations are briefly discussed in Section 5, namely the thermodynamical counterpart (see Murdoch[10]) of the mechanical theory employed in this paper, the extension of the work reviewed to interfaces between pairs of solids (see [10, 17]), and a desirable extension of the theory so as to include couple-stresses. Appended is a section in which we explain our notation concerning the differential geometry of surfaces.

2. MECHANICAL THEORY: BASIC EQUATIONS

We consider a body $\mathscr{B}$ whose physical properties in the neighborhood of its surface are sensibly different from those of its interior. This surface region is modelled as a material surface $\mathscr{S}$; that is, the boundary of $\mathscr{B}$ is regarded as a two-dimensional continuous body in its own right, endowed with a structure which reflects the behavior of the surface region. In particular, the stress localized in the surface region is represented by the *surface stress tensor* Σ. This tensor field has the following interpretation: if γ is a smooth curve in $\mathscr{S}$ with unit normal $\boldsymbol{\nu}$ (at any point of γ the vector $\boldsymbol{\nu}$ lies in the tangent plane to $\mathscr{S}$ and is orthogonal to γ) then $\Sigma\boldsymbol{\nu}$ is the force (per unit length of γ) exerted by that portion of $\mathscr{S}$ into which $\boldsymbol{\nu}$ is directed upon its remaining part. We assume that $\mathscr{S}$ adheres to $\mathscr{B}$ without slipping and that both $\mathscr{B}$ and $\mathscr{S}$ are homogeneous, linearly elastic, and isotropic. Then the basic equations consist of the classical equations (e.g. [18])

$$\left.\begin{aligned} \operatorname{div}\mathbf{T} &= \rho\ddot{\mathbf{u}} \\ \mathbf{T} &= \lambda(tr\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} \\ \mathbf{E} &= 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \end{aligned}\right\} \quad \text{in } \mathscr{B} \tag{1}$$

coupled with the surface equations (Ref. [6])

$$\left.\begin{aligned} \operatorname{div}_{\mathscr{S}}\Sigma &= \mathbf{Tn} + \rho_0\ddot{\mathbf{u}} \\ \Sigma &= \sigma\mathbf{l} + 2(\mu_0 - \sigma)\mathbf{l}\mathsf{E} + (\lambda_0 + \sigma)(tr\mathsf{E})\mathbf{l} + \sigma\nabla_{\mathscr{S}}\mathbf{u} \\ \mathsf{E} &= 1/2(D\mathbf{u} + D\mathbf{u}^T) \end{aligned}\right\} \quad \text{on } \mathscr{S}. \tag{2}$$

Here $\mathbf{T}$, $\mathbf{u}$ and $\mathbf{E}$ denote, respectively, the stress, displacement and strain fields on $\mathscr{B}$; the constants ρ, λ and μ are the mass density and the Lamé moduli for $\mathscr{B}$; Σ and E denote the surface stress and surface strain fields on $\mathscr{S}$; the constants ρ_0, σ, λ_0 and μ_0 are the surface mass density, the residual surface tension and the surface Lamé moduli; $\mathbf{n}$ is the outward unit normal to $\mathscr{S}$. The remaining notation is explained in the Appendix. In writing eqns (1) and (2) we have tacitly assumed that the environment exerts no force on any part of the body.

Some sample values for the moduli are given in the table below for two cases of physical interest. The first column (iron free surface) gives the surface moduli for a freshly cleaved surface: the values of λ_0 and μ_0 are computed using the results of Price and Hirth[19] together with the (somewhat dubious) assumption of isotropy; the density ρ_0 is computed from the bulk value for iron assuming a thickness of 10 Å for the surface[19]. The second column gives the moduli for a 10^3 Å iron film deposited upon a glass substrate: the density and surface moduli are computed using the bulk values, while the residual surface stress σ comes from data of Klokholm and Berry[20].

3. EQUILIBRIUM PROBLEMS

Here we examine time-independent solutions to eqns (1) and (2) in a number of cases. We show that, for situations in which the physical parameters have the orders of magnitude listed in Table 1, static experiments for the determination of the surface Lamé moduli λ_0 and μ_0 will be difficult to devise, since their effect is extremely small.

Table 1.

moduli	iron free surface	10^3 Å iron film on glass substrate
σ (N/m)	1.7	110
μ_0 "	2.5	8×10^3
λ_0 "	-8	7×10^3
ρ_0 (kg/m^2)	7×10^{-6}	7×10^{-4}
μ (N/m^2)	7×10^{10}	2.25×10^{10}
λ "	8×10^{10}	2.25×10^{10}
ρ (kg/m^3)	7×10^3	3×10^3

3.1 *Infinite cylinder, cylindrical hole, and sphere*

Let $\mathcal{B}$ be an infinite circular cylinder of radius a. Then for the corresponding plane problem[6] the stress is a uniform pressure of magnitude

$$\sigma/a(1+\xi),$$

where

$$\xi = (\lambda_0 + 2\mu_0)/2a(\lambda + \mu),$$

and the radius decreases by

$$\sigma/2(\lambda + \mu)(1+\xi).$$

For the two cases listed in Table 1, $\xi a = 10^{-11}$ m and 2.6×10^{-7} m, respectively, so that ξ is negligible compared to unity for any realizable value of a. Setting $\xi = 0$ we see that the radius decreases by the amount

$$\sigma/2(\lambda + \mu),$$

yielding 0.06 Å and 12 Å for the two cases of interest. Note that *this quantity is independent of the radius.* For $\xi = 0$ the pressure is σ/a; even for a as small as 10^{-5} m this pressure is only 1.7×10^5 N/m^2 (25 psi) and 1.1×10^7 N/m^2 (1600 psi).

The companion problem in which $\mathcal{B}$ is the exterior to a cylinder has an equally simple solution[6] with the same qualitative features. It is of interest to note that the cylinder decreases in radius by an amount less than that of the corresponding cylindrical hole.

If $\mathcal{B}$ is a sphere of radius a, then the stress[10] is a uniform pressure of magnitude

$$2\sigma/a(1+\alpha),$$

where

$$\alpha = 2[\sigma + 2(\lambda_0 + \mu_0)]/a(3\lambda + 2\mu).$$

The radius decreases by the amount

$$2\sigma/(3\lambda + 2\mu)(1+\alpha).$$

As in the case of the previous examples, substitution of data from Table 1 reveals that the role played by λ_0 and μ_0 is negligible in the case of the iron free surface, while for an iron film on a glass substrate the radius must be of order 10^{-5} m to influence the calculated value of α by 1%.

3.2 *Square crystal*

Here the problem is plain, the cross section being *square*. This problem was first posed by Herring[3] with $(2)_1$ replaced by the constitutive equation

$$\Sigma = \sigma \mathbf{I}.$$

Since Σ represents the Piola–Kirchhoff stress, rather than the Cauchy stress, this is an *approximative* constitutive equation for the case in which the surface stress is a pure surface tension.

As Herring first noted, for this constitutive equation the loading of the surface on the substrate is equivalent to four concentrated loads, one at each corner of the square, with each load applied diagonally outward (for $\sigma < 0$) and of magnitude $\sqrt{(2)}|\sigma|$. Using a finite element analysis, this problem was solved by Dunham and Gurtin [11] for the following choice of constants:

$$\sigma = -1\ \mathrm{N/m}, \quad \beta = 5 \times 10^{10}\ \mathrm{N/m^2}, \quad \nu = 1/3, \quad l = 10^{-5}\ \mathrm{m}$$

(β = Young's modulus, ν = Poisson's ratio and l = width). The results are shown in Fig. 1. The maximum displacement occurs at the corner and is about 1 Å (this displacement is independent of the choice of l). Although the stress becomes unbounded as the corner is approached, the maximum principal stress at a distance greater than $l/20$ from the corners is less than $20 \times 10^5\ \mathrm{N/m^2}$ (290 psi).

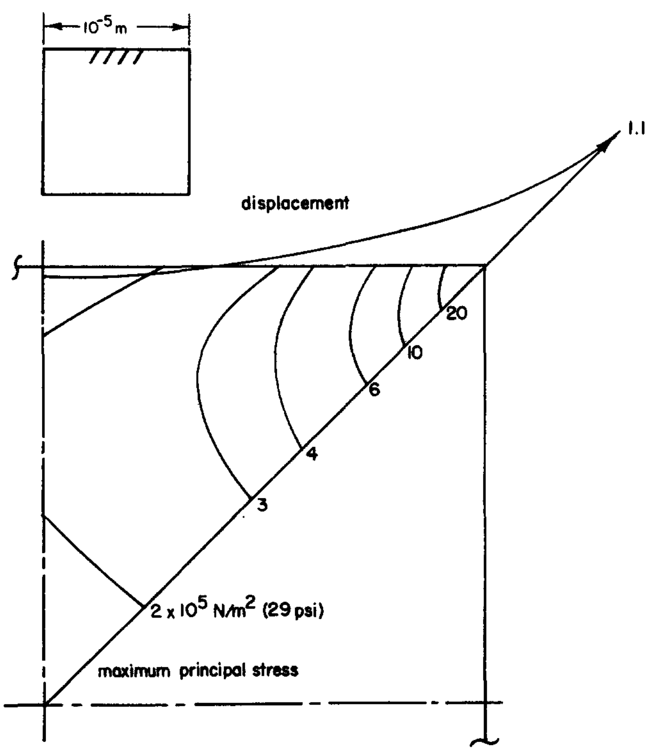

Fig. 1.

3.3 *Surface buckling of a half-space*

Here we consider the plane problem appropriate to a half-space, but now we use the constitutive eqn $(2)_2$ without any approximative assumption. We seek a solution which yields a buckled shape for the free surface, and which decays rapidly with depth. The equations in the interior of the body are easily satisfied with the aid of an Airy stress function (e.g. Gurtin[18]); one with the requisite properties is

$$\varphi(x, y) = (A + By)\, e^{-\omega y} \sin \omega x,$$

where the coordinate system is rectangular with x-axis coincident with the free surface and y-axis pointing into the body. The "boundary condition" $(2)_2$ leads to an eigenvalue problem for the determination of ω and the constants A and B. This problem has a solution with $\omega > 0$ provided either

$$\sigma < 0 \quad \text{or} \quad \lambda_0 + 2\mu_0 < 0.$$

We therefore have the following result of Andreussi and Gurtin[12]: *surface buckling is possible whenever either the residual surface stress is compressive or the surface elastic modulus is negative.*

4. DYNAMICAL PROBLEMS

This section is concerned with the effect of surface stress upon free vibrations of a beam and also upon (plane harmonic) acoustic wave propagation in a half-space.

4.1 *Free vibrations of a beam*

Gurtin, Markenscoff and Thurston[14] have studied the free vibrations of a cantilever within the framework of classical beam theory, but with surface stress included through the boundary condition $(2)_{1-3}$. In particular, they derive the following relation for the lowest natural frequency f:

$$f^2 = f^2_{\text{classical}}\left[1 + \frac{6(\lambda_0 + 2\mu_0)}{\beta h}\right].$$

Here $f_{\text{classical}}$ is the natural frequency neglecting surface stress, h is the thickness of the beam (assumed rectangular in cross section), and β is Young's modulus. Thus, in contrast to the remarks of Lagowski, Gatos and Sproles[13], the residual surface tension σ has *no* effect on the natural frequency. Surface elasticity, however, does: if $\lambda_0 + 2\mu_0$ is negative, as is the case with an iron free surface (see Table 1), then the effect of surface stress will be to *lower* the natural frequency, a result in qualitative agreement with the experimental findings of [13] for GaAs wafers 3–50 μm in thickness, 6–15 mm in length, and 1–1.5 mm in width. However, using the values $\beta = 10^{11}$ N/m^2 and $h = 10^{-5}$ m, the former a reasonable choice for GaAs, together with $\lambda_0 + 2\mu_0 = -3$ N/m, the value for iron which we take as an estimate, we conclude that the correction term is of order 10^{-5} and consequently negligible. Thus, unless the surface moduli of GaAs are 4 or 5 orders of magnitude larger than those of iron, the effect of the free surface (within the experimental range of [13]) is negligible.

On the other hand, for a glass beam of thickness 10^{-5} m whose free surfaces are coated with a 10^3 Å iron film the data in Table 1 yield a correction factor of 0.25. This therefore furnishes a possible method of determining the surface moduli of thin films.

4.2 *Acoustic wave propagation in half-spaces*

Let $\mathscr{B}$ be a body which occupies the half-space $\{\mathbf{x}: x_3 \geq 0\}$, where (x_1, x_2, x_3) denote Cartesian coordinates. Then eqns (1) may be expressed in the form[18]

$$\left.\begin{aligned} T_{ij,j} &= \rho \ddot{u}_i \\ T_{ij} &= \delta_{ij}\lambda u_{k,k} + \mu(u_{i,j} + u_{j,i}) \end{aligned}\right\} \quad \text{for } x_3 \geq 0, \tag{3}$$

where i, j, k range over the integers 1, 2 and 3, summation convention is used, δ_{ij} designates the Kronecker delta, and a subscript preceded by a comma indicates differentiation with respect to the corresponding coordinate. Equations (2) become, in this co-ordinate system (see Appendix),

$$\begin{gathered}\Sigma_{i\alpha,\alpha} + T_{i3} = \rho_0 \ddot{u}_i \\ \Sigma_{\alpha\beta} = \sigma\delta_{\alpha\beta} + (\mu_0 - \sigma)(u_{\alpha,\beta} + u_{\beta,\alpha}) + (\lambda_0 + \sigma)u_{\gamma,\gamma}\delta_{\alpha\beta} + \sigma u_{\alpha,\beta} \quad \text{on } x_3 = 0, \\ \Sigma_{3\beta} = \sigma u_{3,\beta}\end{gathered} \tag{4}$$

where α, β, γ range over the integers 1 and 2.

Equations (4), in the absence of residual stress (that is, when $\sigma = 0$), may be identified with an approximative boundary condition derived by Mindlin[21] in connection with the vibration of plates and utilized by Tiersten[22] in a study of wave propagation in a linearly elastic half-space upon which a thin stratum of another linearly elastic material adheres without slipping. If this stratum is of thickness h and has bulk Lamé moduli λ' and μ' and bulk density ρ', then the Mindlin–Tiersten boundary condition coincides with (4) upon setting $\sigma = 0$ and making the associations

$$(\rho_0, \lambda_0, \mu_0) \longleftrightarrow (\rho' h, \mu' h, 2\lambda'\mu' h/(\lambda' + \mu')). \tag{5}$$

In particular, ρ_0 and μ_0 are merely scaled versions of their bulk counterparts. The Mindlin–Tiersten boundary condition was shown to be valid provided that any vibrations to which it is applied have wavelengths markedly in excess of the plate/stratum thickness. Hence the identifications (5) relate our theory to three-dimensional considerations and provide a guide to its range of validity. This is particularly important since, as will be seen, the surface effects become most pronounced at high frequencies.

Bulk waves

Solutions to eqns (3) which represent plane harmonic waves may be categorized[23] as transverse (shear or S-waves) or longitudinal (pressure or P-waves). The classical results (corresponding to the boundary condition $T_{i3} = 0$ on $x_3 = 0$) are modified by the inclusion of surface stress and inertia via the required satisfaction of eqns (4). For example, consider an SH wave with angle of incidence α propagating in the 1–3 plane with amplitude parallel to the 2-axis. Then, as in the classical theory without surface stress, the reflected wave is also an SH wave with angle of reflection equal to α (Fig. 2) and amplitude equal to the amplitude of the incident wave (which we take equal to 1). Thus

$$\begin{gathered}u_1 = u_3 = 0, \\ u_2 = \cos\left\{k\left(\frac{x_I}{b} - t\right)\right\} + \cos\left\{k\left(\frac{x_R}{b} - t\right) - \delta\right\},\end{gathered}$$

where k is the frequency, $b > 0$ is the speed of shear waves in $\mathcal{B}$ ($b^2 = \mu/\rho$), and

$$x_I = x_1 \sin\alpha - x_3 \cos\alpha, \quad x_R = x_1 \sin\alpha + x_3 \cos\alpha.$$

The only departure from the classical theory lies in the presence of the *phase angle* δ, which is given by

$$\delta = 2\tan^{-1}\kappa$$

with

$$\kappa = \left\{\frac{\eta\sin^2\alpha - 1}{\cos\alpha}\right\}\frac{k}{k_0}, \quad \eta = \frac{\mu_0}{\rho_0 b^2}, \quad k_0 = \frac{\rho b}{\rho_0}.$$

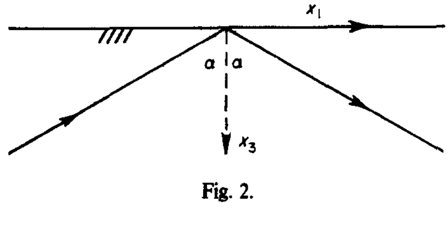

Fig. 2.

Since the phase angle δ depends upon frequency, a linear combination of *SH* waves of different frequencies will, upon reflection, yield a distorted wave. It is interesting to note that of the surface moduli only μ_0 and ρ_0 are involved; in particular, no role is played by the residual stress σ.

We now turn back to the two cases of interest specified in Table 1; for these the values of η and $k_0/2\pi$ are listed below:

Table 2.

	iron half-space	10^3 Å iron film on glass substrate
η	.036	1.5
$k_0/2\pi$	5×10^{11} Hz	1.9×10^{9} Hz

The number $k_0/2\pi$ represents a characteristic frequency. The results for the iron half-space suggest that surface effects become important at frequencies far too high to make wave propagation a practical means of measuring surface parameters. However, the frequency of 1.9×10^9 Hz is experimentally attainable, while at the same time (since it corresponds to a wavelength of 1.5×10^{-6} m) lying in a domain for which the surface model is valid.

The reflection of incident *SV* and *P* waves at a plane boundary has also been discussed in detail[15], the departure from classical behavior in these cases resulting in the appearance of frequency-dependent phase angles and amplitudes for the reflected waves. It follows that a superposition of any of the bulk waves considered will suffer distortion (but not dispersion) upon reflection at a plane boundary which supports surface stress.

Surface waves [16]

The effect of the boundary condition (4) upon the propagation of (Rayleigh) surface waves[23] is to introduce dispersion. That is, the wave-speed is a function of its frequency, in marked contrast to the classical situation in which waves of all frequencies propagate at a single fixed speed. The precise behavior depends upon the relative values of the parameters σ, $\lambda_0 + 2\mu_0$, and $\rho_0 b^2$, where b is the (bulk) shear wave-speed. Using the data in Table 1 for a thin iron film upon a glass substrate, it becomes apparent that experiments involving high-frequency surface waves may prove practicable in determining the values of surface parameters: frequencies within the range of possible generation, and for which the surface model is appropriate, result in significant dispersion.

Love waves[23] also exist whenever $\mu_0/\rho_0 < \mu/\rho$ and are dispersive, the wave-speed decreasing with frequency from a maximum of $(\mu/\rho)^{1/2}$ to an asymptotic minimum of $(\mu_0/\rho_0)^{1/2}$. Thus such waves are possible for an iron free surface because

$$\mu_0/\rho_0 = 0.36 \times 10^6 < \mu/\rho = 10^7$$

but impossible for the iron film upon glass substrate, since

$$\mu_0/\rho_0 = 1.1 \times 10^7 > \mu/\rho = 0.75 \times 10^7$$

in this case. However, it must be remembered that ρ_0 in this latter case was taken to be a scaled version of the bulk density, an assumption which is probably not justified. Indeed, there is evidence that the film density is smaller than the bulk density times the thickness (see Chopra[24]).

5. FURTHER CONSIDERATIONS

A complete thermodynamical theory of bodies whose boundaries are elastic material surfaces has been developed by Murdoch[10]. As pointed out by Shuttleworth[2] and

Herring[4], the surface stress is not in general numerically equal to the (Helmholtz) free energy per unit (surface) area. Indeed, for an *elastic* material surface, this is the case only if this free energy is independent of deformations of the surface. An interesting feature of this theory is its prediction of the dependence of the coefficient of thermal expansion of a given material upon specimen size, this having its origin in surface thermo-elasticity.

As evidenced by the relaxation of surface stress in crystals cleaved *in vacuo* when subsequently exposed to a gaseous environment[1], surface stress is essentially interfacial in character. Moreover, when two solid bodies adhere without slipping then interfacial stress is to be expected in the neighborhood of their common boundary. Indeed, in practical situations this adhesion must be effected by a bonding process of some kind which must be expected to result in an interfacial region whose material properties are sensibly different from those of either body and yet which is small in the dimension separating the bodies. If the interfacial region is modelled as an elastic material surface then the linearized mechanical theory is given by two sets of equations for the bulk continua, each having the form of eqns (1), together with a set of equations identical to (2) except for the inclusion in eqn $(2)_1$ of a term representing the traction upon the interface exerted by the second body. Interfacial wave propagation along a plane boundary between two semi-infinite linearly elastic bodies has been discussed[17]. The results resemble those for the surface waves described in Section 4 and demonstrate the possibility of propagating Stoneley waves[23] for nearly all pairs of continua, in contrast to the classical theory in which these bodies must have similar acoustical properties.

In the work discussed in this paper the surface or interfacial region is modelled as a membrane[25]; that is, a material surface which offers no resistance to bending. This is a reasonable assumption with which to begin a theoretical investigation of such regions from a continuum viewpoint, but it would clearly be preferable to set the membrane model within a more general framework when this is used to describe the behavior of thin films and surfaces of peened bodies for which the thicknesses involved are considerably greater than those of surface regions of single continua. That is, surfaces with a structure which allows them to support couple-stresses (or bending moments) should be studied. Such an investigation should delineate the range of validity of the membrane model in these cases and also throw light upon the oft-asserted (constitutive) dependence of surface stress upon curvature (of course, the *effect* of any surface or interfacial stress may involve curvature through the appearance in eqn $(2)_1$ of the surface divergence of the surface or interfacial stress tensor).

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APPENDIX

The purpose of this appendix is to introduce, briefly, our notation and terminology. For a more thorough discussion the reader is referred to Ref. [6].

Let $\mathscr{E}$ denote a three-dimensional Euclidean point space, with $\mathscr{V}$ the corresponding vector space. Further, let $\mathscr{S}$ be a smooth surface in $\mathscr{E}$, and for each $\mathbf{x} \in \mathscr{S}$ let $\mathrm{T}_\mathbf{x}$ denote the tangent plane to $\mathscr{S}$ at $\mathbf{x}$. The tangent space $\mathscr{T}_\mathbf{x}$ is the (two-dimensional) vector space of displacements of points in $\mathrm{T}_\mathbf{x}$ from $\mathbf{x}$. Locally, $\mathscr{S}$ may be characterized by the following bijective correspondence between sufficiently small vectors $\mathbf{w} \in \mathscr{T}_\mathbf{x}$ and points of $\mathscr{S}$: let $\pi_\mathbf{x}(\mathbf{w})$ represent the point of $\mathscr{S}$ whose perpendicular projection upon $\mathrm{T}_\mathbf{x}$ is the point $\mathbf{x}+\mathbf{w}$. Thus $\pi_\mathbf{x}$ is defined on a neighborhood of the zero vector in $\mathscr{T}_\mathbf{x}$ and

$$\pi_\mathbf{x}(\mathbf{w}) = \mathbf{x} + \mathbf{w} + \mathbf{v}, \tag{6}$$

where $\mathbf{v} \in \mathscr{T}_\mathbf{x}^{\perp}$, the (one-dimensional) subspace of V orthogonal to $\mathscr{T}_\mathbf{x}$. Further, the tangency condition implies that

$$\mathbf{v} = o(\mathbf{w}) \quad \text{as} \quad \mathbf{w} \to \mathbf{0}. \tag{7}$$

Since $\pi_\mathbf{x}(\mathbf{0}) = \mathbf{x}$, eqns (6) and (7) yield

$$\pi_\mathbf{x}(\mathbf{w}) = \pi_\mathbf{x}(\mathbf{0}) + \mathbf{w} + o(\mathbf{w}) \quad \text{as} \quad \mathbf{w} \to \mathbf{0},$$

so that

$$(\nabla \pi_\mathbf{x})(\mathbf{0}) = \mathbf{I}(\mathbf{x}),$$

where

$$\mathbf{I}(\mathbf{x})\mathbf{w} = \mathbf{w} \quad \text{for all} \quad \mathbf{w} \in \mathscr{T}_\mathbf{x}.$$

That is, $\mathbf{I}(\mathbf{x})$ is the *inclusion mapping* of $\mathscr{T}_\mathbf{x}$ into $\mathscr{V}$. More generally, if $\mathbf{u}$ is a point-or vector-valued function (so that $\mathbf{u} \circ \pi_\mathbf{x}$ is defined in a neighborhood of $\mathbf{0}$ in $\mathscr{T}_\mathbf{x}$), then $\mathbf{u}$ is said to be differentiable at $\mathbf{x}$ if $\mathbf{u} \circ \pi_\mathbf{x}$ is differentiable at $\mathbf{0}$. In this case, the *surface gradient* of $\mathbf{u}$ at $\mathbf{x}$, $(\nabla_\mathscr{S}\mathbf{u})(\mathbf{x})$, is defined by

$$(\nabla_\mathscr{S}\mathbf{u})(\mathbf{x}) = (\nabla(\mathbf{u} \circ \pi_\mathbf{x}))(\mathbf{0}),$$

where $(\nabla(\mathbf{u} \circ \pi_\mathbf{x}))(\mathbf{0})$ denotes the ordinary gradient of $\mathbf{u} \circ \pi_\mathbf{x}$ evaluated at $\mathbf{0} \in \mathscr{T}_\mathbf{x}$. It follows that $(\nabla_\mathscr{S}\mathbf{u})(\mathbf{x})$ is a linear mapping from $\mathscr{T}_\mathbf{x}$ into $\mathscr{V}$.

Let $\mathscr{S}$ be oriented by a choice of smooth unit normal field $\mathbf{n}$ (in applications $\mathscr{S}$ is the boundary of a region $\mathscr{B}$ and $\mathbf{n}$ is the outward unit normal to $\mathscr{S}$). Then

$$\mathbf{P} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}$$

is the projection onto the tangent space; that is, $\mathbf{P}(\mathbf{x})$ maps each vector $\mathbf{v} \in \mathscr{V}$ into its component in the tangent space $\mathscr{T}_\mathbf{x}$. We call

$$D\mathbf{u} = \mathbf{P}\nabla_\mathscr{S}\mathbf{u}$$

the *tangential derivative* of $\mathbf{u}$. $D\mathbf{u}(\mathbf{x})$, at any given $\mathbf{x}$, maps tangent vectors into tangent vectors; thus its transpose $D\mathbf{u}(\mathbf{x})^T$ and trace $tr(D\mathbf{u}(\mathbf{x}))$ are well-defined.

If $\mathbf{u}$ is a vector field on $\mathscr{S}$ and Σ is a tensor field on $\mathscr{S}$ with $\Sigma(\mathbf{x})$: $\mathscr{T}_\mathbf{x} \to \mathscr{V}$, then the *surface divergence* $\mathrm{div}_\mathscr{S}\, \mathbf{u}$ of $\mathbf{u}$ is defined by

$$\mathrm{div}_\mathscr{S}\, \mathbf{u} = tr(D\mathbf{u}),$$

while the surface divergence $\mathrm{div}_\mathscr{S}\, \Sigma$ of Σ is that vector field which satisfies†

$$(\mathrm{div}_\mathscr{S}\, \Sigma) \cdot \mathbf{k} = \mathrm{div}_\mathscr{S}\,(\Sigma^T \mathbf{k})$$

for every fixed vector $k \in \mathscr{V}$.

When $\mathscr{S}$ is a plane surface the foregoing becomes particularly simple since, for every $\mathbf{x} \in \mathscr{S}$, $\mathscr{T}_\mathbf{x}$ is the vector space $\mathscr{T}$ of all displacements in this plane, and $\pi_\mathbf{x}$ associates with every vector $\mathbf{t} \in \mathscr{T}$ the point $\mathbf{x}+\mathbf{t}$. Thus, if $\mathbf{r}$ is a vector field defined on $\mathscr{S}$, $\mathbf{e}$ is any unit vector in $\mathscr{T}$, and s is a real number, then

$$\begin{aligned} ((\nabla_\mathscr{S}\mathbf{u})(\mathbf{x}))s\mathbf{e} &= (\mathbf{u} \circ \pi_\mathbf{x})(s\mathbf{e}) - (\mathbf{u} \circ \pi_\mathbf{x})(\mathbf{0}) + o(s) \\ &= \mathbf{u}(\mathbf{x} + s\mathbf{e}) - \mathbf{u}(\mathbf{x}) + o(s). \end{aligned}$$

Dividing by s and proceeding to the limit as $s \to 0$,

$$(\nabla_\mathscr{S}\mathbf{u})\mathbf{e} = \partial\mathbf{u}/\partial s.$$

That is, the action of $\nabla_\mathscr{S}\mathbf{u}$ upon any unit vector $\mathbf{e}$ delivers the rate of change of $\mathbf{u}$ in the direction defined by $\mathbf{e}$. Hence, if $\mathbf{e}_1$,

†Σ^T: $\mathscr{V} \to \mathscr{T}_\mathbf{x}$ is defined by $\Sigma^T\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \Sigma\mathbf{w}$ for all $\mathbf{v} \in \mathscr{V}$ and $\mathbf{w} \in \mathscr{T}_\mathbf{x}$.

$\mathbf{e}_2$, $\mathbf{e}_3$ form an orthonormal basis for $\mathscr{V}$ with $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathscr{T}$, then

$$(\nabla_{\mathscr{S}}\mathbf{u})_{i\beta} = \mathbf{e}_i \cdot (\nabla_{\mathscr{S}}\mathbf{u})\,\mathbf{e}_\beta = u_{i,\beta} \quad (i = 1, 2, 3;\ \beta = 1, 2),$$

where $u_{i,\beta}$ is the rate of change of u_i $(=\mathbf{u}\cdot\mathbf{e}_i)$ in the direction of $\mathbf{e}_\beta$. Upon noting that in this case,

$$\mathbf{P} = \mathbf{1} - \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2,$$

$$(D\mathbf{u})_{\alpha\beta} = u_{\alpha,\beta} \quad \text{and} \quad (\mathbf{E})_{\alpha\beta} = 1/2(u_{\alpha,\beta} + u_{\beta,\alpha}),$$

it follows that eqn $(2)_2$ reduces to $(4)_{2,3}$ when $\mathscr{S}$ is plane. Further,

$$(\nabla_{\mathscr{S}}(\Sigma_{i\alpha}\mathbf{e}_\alpha))\mathbf{e}_\beta = (\Sigma_{i\alpha}\mathbf{e}_\alpha)_{,\beta} = \Sigma_{i\alpha,\beta}\mathbf{e}_\alpha,$$

whence

$$(D(\Sigma_{i\alpha}\mathbf{e}_\alpha))\mathbf{e}_\beta = \Sigma_{i\alpha,\beta}\mathbf{e}_\alpha$$

and thus

$$\begin{aligned}(\mathrm{div}_{\mathscr{S}}\,\Sigma)_i = (\mathrm{div}_{\mathscr{S}}\,\Sigma)\cdot\mathbf{e}_i = \mathrm{div}_{\mathscr{S}}\,(\Sigma^T\mathbf{e}_i) &= \mathrm{div}_{\mathscr{S}}\,(\Sigma_{i\alpha}\mathbf{e}_\alpha) \\ &= tr(D(\Sigma_{i\alpha}\mathbf{e}_\alpha)) = \Sigma_{i\beta,\beta},\end{aligned}$$

so explaining the derivation of the first term in eqn $(4)_1$.